A Practical Guide to Total Differentiation

Look at the following model for a closed economy in the long run:

1) $Y = C + I + \overline{G}$
2) $C = C(Y - \overline{T})$
3) $I = I(r)$
4) $Y = F(\overline{K}, \overline{L})$

Imagine, for example, that we want to find the effect of expansive financial policy on GDP, $\frac{\partial Y}{\partial \overline{G}}$. In other circumstances we might find this by solving by substitution to find *Y*. We could, for example, insert 2) and 3) in 1) to get $Y = C(Y - \overline{T}) + I(r) + \overline{G}$. But we can no longer isolate *Y*! What do we do? We are saved by the following mathematical theorem:

If *y* can be written as $y = f(x)$

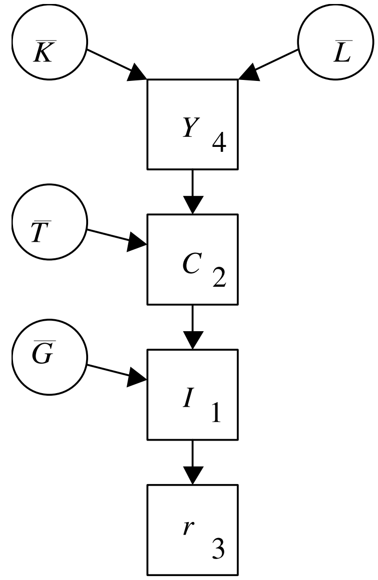
$\Rightarrow$ changes in *y* can be written as $dy \approx f'(x)dx$ for small *dx*.

Intuitively, we can see the sense in this by dividing by *dx* on both sides of the final equation to give $\frac{dy}{dx} \approx f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial y}{\partial x}$, which is of course quite logical. The change in *y* when we change *x* is approximately the same as the derivative of *y* with respect to *x* which is defined as the change in *y* for marginal (very small) changes in *x*.

This may sound like gobbledegook, and a real mathematician would probably tell me that it wasn't even precisely correct, but as macroeconomists we only need to understand its practical uses. We can see this by totally differentiating the model described above. When doing this we must remember that we must differentiate with respect to *all* of the variables in each of the equations, using the method described in the theorem. The result (through repeated use of the chain rule) is as follows:

1) $\Rightarrow dY = dC + dI + d\overline{G}$
2) $\Rightarrow dC = C'(Y - \overline{T})dY - C'(Y - \overline{T})d\overline{T}$
3) $\Rightarrow dI = I'(r)dr$
4) $\Rightarrow dY = F_{\overline{K}}'(\overline{K}, \overline{L})d\overline{K} + F_{\overline{L}}'(\overline{K}, \overline{L})d\overline{L}$

We now turn to the arrow diagram for the model, which is given here. We stick to our original example of a change in $\overline{G}$. How does this affect the various variables? It is obvious from the arrow diagram that there is no impact on *Y* or *C*. The same can be said for the exogenous variables. So we know that $dY = dC = d\overline{K} = d\overline{L} = d\overline{T} = 0$. It goes without saying that $d\overline{G} \neq 0$, since this is the variable we are adjusting!



So the only variables we need to look at are *I* and *r*. We can see from the arrow diagram that we need first to look at *dI* in equation 1. From this we get

$dI = dY - dC - d\overline{G}$.

We know that $dY = dC = 0$, so we have

$dI = -d\overline{G}$.

Now we simply divide by $d\overline{G}$ on both sides of the equation to get our result:

$$\frac{dI}{d\overline{G}} = -1.$$

In other words, there is full crowding out of investment.

The arrow diagram now tells us that we need to look at *dr* in equation 3. We start by isolating *dr*:

$$dr = \frac{1}{I'(r)} dI .$$

We now use the same trick as before, and simply divide by $d\overline{G}$ to get

$$\frac{dr}{d\overline{G}} = \frac{1}{I'(r)} \frac{dI}{d\overline{G}} .$$

Now you can see why it was important that we used the order given in the arrow diagram. We have already found $dI/d\overline{G} = -1$, and we can insert this to get the result:

$$\frac{dr}{d\overline{G}} = -\frac{1}{I'(r)} .$$

This is positive, since $I'(r) < 0$.

So, without solving the model by substitution, we have reached our usual conclusions – that in a closed economy in the long run, a rise in government consumption leads to a fall in savings, which results in a rise in interest rates and full crowding-out of investment.

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